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K_3-WORM colorings of graphs: Lower chromatic number and gaps in the chromatic spectrum *

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Abstract

A K_3-WORM coloring of a graph G is an assignment of colors to the vertices in such a way that the vertices of each K_3-subgraph of G get precisely two colors. We study graphs G which admit at least one such coloring. We disprove a conjecture of Goddard et al. [*Congr. Numer.*, 219 (2014) 161–173] who asked whether every such graph has a K_3-WORM coloring with two colors. In fact for every integer $k \geq 3$ there exists a K_3-WORM colorable graph in which the minimum number of colors is exactly k. There also exist K_3-WORM colorable graphs which have a K_3-WORM coloring with two colors and also with k colors but no coloring with any of $3, \ldots, k-1$ colors. We also prove that it is NP-hard to determine the minimum number of colors and NP-complete to decide k-colorability for every $k \geq 2$ (and remains intractable even for graphs of maximum degree 9 if $k = 3$). On the other hand, we prove positive results for d-degenerate graphs with small d, also including planar graphs. Moreover we point out a fundamental connection with the theory of the colorings of mixed hypergraphs. We list many open problems at the end.

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1 Introduction

In a vertex-colored graph, a subgraph is *monochromatic* if its vertices have the same color; and it is *rainbow* if its vertices have pairwise different colors. Given two graphs F and G, an F*-WORM coloring* of G is an assignment of colors to its vertices such that no subgraph isomorphic to F is either monochromatic or rainbow. This notion was introduced recently in [8] by Goddard, Wash, and Xu. As noted in [8], however, for some types of F some earlier results due to Bujtás et al. [3, 4] imply upper bounds on the possible number of colors in F-WORM colorings of graphs G. The name "F-WORM" comes as the abbreviation of "WithOut a Rainbow or Monochromatic subgraph isomorphic to F".

If G has at least one F-WORM coloring, then $W^-(G,F)$ denotes the minimum number of colors and $W^+(G,F)$ denotes the maximum number of colors in an F-WORM coloring of G; they are termed the F-WORM *lower* and *upper chromatic number*, respectively. Moreover, the F-WORM *feasible set* $\Phi_{_W}(G,F)$ of G is the set of those integers s for which G admits an F-WORM coloring with exactly s colors. In general, we say that G has a *gap* at k in its F-WORM chromatic spectrum, if $W^-(G,F) < k < W^+(G,F)$ but G has no F-WORM coloring with precisely k colors. Otherwise, if $\Phi_{_W}(G,F)$ contains all integers between $W^-(G,F)$ and $W^+(G,F)$, we say that the F-WORM feasible set (or the F-WORM chromatic spectrum) of G is *gap-free*.

We shall not mention later in each assertion, but it should be emphasized that the values $W^-(G,F)$ and $W^+(G,F)$ are defined only for F-WORM-colorable graphs. Hence, wherever W^- or W^+ appears in the text, it is assumed that the graph in question is colorable.

As one can see, four fundamental problems arise in this context: testing whether G is F-WORM colorable, computing $W^-(G,F)$, computing $W^+(G,F)$, and determining $\Phi_{_W}(G,F)$.

1.1 Results

In this paper we focus on the case of $F = K_3$, i.e. K_3-WORM colorings of graphs. It is clear that K_5 has no K_3-WORM coloring. Moreover, $W^-(G,K_3) = 1$ and $W^+(G,K_3) = n$ are valid for all triangle-free n-vertex graphs G (and only for them), and any number of colors between 1 and n can occur in this case. Therefore, the interesting examples are the graphs whose clique number equals 3 or 4.

Goddard, Wash, and Xu [7] proved that $W^-(G, K_3) \leq 2$ holds for outerplanar graphs and also for cubic graphs. They conjectured that every K_3-WORM-colorable graph admits a K_3-WORM coloring with two colors ([7, Conjecture 1]). Our Theorem 3 disproves this conjecture in a wide sense, showing that the minimum number of colors in K_3-WORM-colorable graphs can be arbitrarily large. The conjecture is false even in the class of K_4-free graphs, as demonstrated by our Theorem 5.

It was proved in [8] that there exist graphs with gaps in their P_3-WORM chromatic spectrum. In [7], the authors remark that for trees the K_3-WORM chromatic spectrum is trivially gap-free (as noted above, it is clearly so for all triangle-free graphs), and they ask whether this is true for every K_3-WORM colorable graph. Our constructions presented in Section 3 show the existence of graphs H_k which have $W^-(H_k, K_3) = 2$ and $W^+(H_k, K_3) \geq k$, but the feasible set $\Phi_W(G, K_3)$ contains no element from the range $[3, k-1]$. Further types of constructions (applying a different kind of methodology) and a study of the K_3-WORM upper chromatic number will be presented in our follow-up paper [6].

Goddard, Wash, and Xu proved that the decision problem whether a generic input graph admits a K_3-WORM coloring is NP-complete ([7, Theorem 3]). We consider complexity issues related to the determination of $W^-(G, K_3)$. In Section 5, we show that it is NP-hard to distinguish between graphs which are K_3-WORM-colorable with three colors and those needing precisely four as minimum. This hardness is true already on the class of graphs with maximum degree 9. Additionally, we prove that for every $k \geq 4$, the decision problem wether $W^-(G, K_3) \leq k$ is NP-complete already when restricted to graphs with a sufficiently large but bounded maximum degree. Deciding K_3-WORM 2-colorability is hard, too, but so far we do not have a bounded-degree version of this result. We also prove that the algorithmic problem of deciding if the K_3-WORM chromatic spectrum is gap-free is intractable.

A graph is *3-degenerate* if each of its non-empty subgraphs contains a vertex of degree at most 3. In Section 6 we point out that every 3-degenerate graph is K_3-WORM-colorable with two colors and has a gap-free K_3-WORM chromatic spectrum. For graphs of maximum degree 3, a formula for $W^+(G, K_3)$ can also be given. It was observed by Ozeki [15] that the property of being K_3-WORM 2-colorable is also valid for planar graphs. It extends to 4-colorable graphs as a common generalization of the two graph classes just mentioned. The corresponding theorem is stated at the end of

Section 6; we thank Kenta Ozeki for kindly allowing us to include the result here.

We conclude the paper with several open problems and conjectures in Section 7.

2 Mixed bi-hypergraphs

The notion of mixed hypergraph was introduced by Voloshin in the 1990s [16, 17]. A detailed overview of the theory is given in the monograph [18]. Many open problems in the area are surveyed in [2]. In the present context the relevant structures will be what are called 'mixed bi-hypergraphs'.[1]

In general, a *mixed hypergraph* $\mathcal{H}$ is a triplet $(X, \mathcal{C}, \mathcal{D})$, where X is the vertex set and $\mathcal{C}$ and $\mathcal{D}$ are set systems over X. A coloring of $\mathcal{H}$ is a mapping

$$\varphi : X \to \mathbb{N}$$

with the following two properties:

(c) every set $C \in \mathcal{C}$ contains two vertices with a common color;

(d) every set $D \in \mathcal{D}$ contains two vertices with distinct colors.

A *mixed bi-hypergraph* is a mixed hypergraph with $\mathcal{C} = \mathcal{D}$.

Note that assuming $\mathcal{C} = \varnothing$ the condition (d) just means proper coloring in the usual sense, whereas assuming $\mathcal{D} = \varnothing$ the condition (c) leads to the notion called C-coloring, whose literature has been surveyed in [5].

For a given mixed hypergraph $\mathcal{H}$, four fundamental questions arise in a very natural way.

Colorability. Does $\mathcal{H}$ admit any coloring?

Lower chromatic number. If $\mathcal{H}$ is colorable, what is the minimum number $\chi(\mathcal{H})$ of colors in a coloring?

Upper chromatic number. If $\mathcal{H}$ is colorable, what is the maximum number $\overline{\chi}(\mathcal{H})$ of colors in a coloring?

[1] In the literature of mixed hypergraphs the term simply is 'bi-hypergraph'. Since here our main subject is a different structure class, we will emphasize that it is a *mixed* bi-hypergraph.

Feasible set. If $\mathcal{H}$ is colorable, what is the set $\Phi(\mathcal{H})$ of integers s such that $\mathcal{H}$ admits a coloring with exactly s colors?

The next observation shows that mixed hypergraph theory provides a proper and very natural general framework for the study of F-WORM colorings.

Proposition 1 *Let F be a given graph. For any graph G on a vertex set V, let $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ be the mixed bi-hypergraph in which $X = V$, and both $\mathcal{C}$ and $\mathcal{D}$ consist of those vertex subsets of cardinality $|V(F)|$ which induce a subgraph containing F in G. Then:*

(i) G is F-WORM-colorable if and only if $\mathcal{H}$ is colorable.

(ii) $W^-(G, F) = \chi(\mathcal{H})$.

(iii) $W^+(G, F) = \overline{\chi}(\mathcal{H})$.

(iv) $\Phi_{_W}(G, F) = \Phi(\mathcal{H})$.

Proof. By the definitions, an assignment $\varphi : V \to \mathbb{N}$ is an F-WORM coloring of G if and only if it is a feasible coloring of the mixed bi-hypergraph $\mathcal{H}$. Then, the statements $(i) - (iv)$ immediately follow. $\square$

A similar bijection between 'WORM edge colorings' of K_n and the colorings of a mixed bi-hypergraph defined in a suitable way on the edge set of K_n was observed by Voloshin in e-mail correspondence to us in 2013 [19].

Due to the strong correspondence above, it is meaningful and reasonable to adopt the terminology of mixed hypergraphs to the study of WORM colorings.

3 Large W^- and gap in the chromatic spectrum

We start with a connected triangle-free graph G_k whose chromatic number is equal to k. It is well-known for over a half century, by explicit constructions and also by applying the probabilistic method, that such graphs exist; see

e.g. [9, Section 1.5] for references. We denote the vertices of G_k by $v_1, \ldots, v_n$. Let us consider the graph

$$H_k := G_k \boxtimes K_2$$

which is obtained from G_k by replacing each vertex v_i with two adjacent vertices x_i, y_i and each edge v_iv_j with a copy of K_4 on the vertex set $\{x_i, y_i, x_j, y_j\}$.

Lemma 2 *Let G_k be a connected and triangle-free graph whose chromatic number equals k, for some integer $k \geq 2$. Then the graph H_k defined as $G_k \boxtimes K_2$ has the following properties.*

- (i) $W^-(H_k, K_3) = 2$.
- (ii) *The K_3-WORM colorings of H_k with any number $s \geq k$ of colors, in which x_1 and y_1 get the same color, are in one-to-one correspondence with the proper vertex colorings of G_k with s colors.*
- (iii) *If $k \geq 4$ and $3 \leq t \leq k-1$, then H_k admits no K_3-WORM colorings with t colors.*

Proof. Since G_k is triangle-free, each triangle of H_k is inside a copy of K_4 originating from an edge of G_k. Thus, the K_3-WORM colorings of H_k are precisely those vertex colorings in which

- $(*)$ each copy K of K_4 gets exactly two colors such that each of them appears on exactly two vertices of this K.

For this reason, a K_3-WORM coloring is easily obtained by assigning color 1 to all vertices x_i and color 2 to all vertices y_i. This proves (i).

If x_1 and y_1 have the same color, and a vertex v_j is adjacent to v_1, then, by $(*)$, the only way in a K_3-WORM coloring is to assign x_j and y_j to the same color which is different from the color of $\{x_1, y_1\}$. This property of monochromatic pairs propagates along paths, therefore each pair $\{x_i, y_i\}$ $(1 \leq i \leq n)$ is monochromatic whenever G is connected. Assigning the color of $\{x_i, y_i\}$ to vertex v_i yields a proper vertex coloring of G, and vice versa.

On the other hand, if x_1 and y_1 have distinct colors, and a vertex v_j is adjacent to v_1, then again by $(*)$, the only way in a K_3-WORM coloring is to assign $\{x_j, y_j\}$ to the same pair of colors. Consequently, under the assumption that G is connected, if the colors of x_1 and y_1 are different then precisely two colors are used in the entire graph; and if the colors of x_1 and y_1 are the same then at least k colors occur. This completes the proof of the lemma. □

Theorem 3 *For every $k \geq 3$ there exists a graph F_k such that $W^-(F_k, K_3) = k$.*

Proof. Start with a triangle-free and connected graph G_k whose chromatic number is k, and let H_k be again $G_k \boxtimes K_2$, as above. Let F_k be the graph obtained from three vertex-disjoint copies H_k^i of H_k ($i = 1, 2, 3$) by the following three identifications of vertices:

$$x_1^1 = y_1^2, \qquad x_1^2 = y_1^3, \qquad x_1^3 = y_1^1.$$

This graph is K_3-WORM-colored if and only if so is each H_k^i and moreover the triangle $\{x_1^1, x_1^2, x_1^3\}$ gets precisely two colors.

Suppose, without loss of generality, that x_1^1 and x_1^2 get color 1, and x_1^3 gets color 2. Then, according to Lemma 2, both H_k^1 and H_k^3 are colored entirely with $\{1, 2\}$. On the other hand, we have $\{x_1^2, y_1^2\} = \{x_1^1, x_1^2\}$, hence this vertex pair is monochromatic in color 1, therefore H_k^2 is colored according to a proper vertex coloring of H_k. Thus, the smallest possible number of colors equals the chromatic number k of G_k. $\square$

Theorem 4 *The feasible sets of K_3-WORM-colorable graphs may contain arbitrarily large gaps.*

Proof. In fact this was already proved in part *(iii)* of Lemma 2 which yields a graph H_k whose feasible set is

$$\{2\} \cup \left\{ s \mid k \leq s \leq \frac{|V(H_k)|}{2} \right\}.$$

$\square$

4 K_4-free graphs

Here we prove that $W^-(G, K_3)$ can exceed 2 even when K_4 is not a subgraph of G.

Theorem 5 *There exist K_4-free graphs G such that $W^-(G, K_3) = 3$.*

Proof. We present a construction built in a few steps.

Step 1: Hypergraphs.

It was proved by Lovász [11] that, for every three integers $k, r, g \geq 3$ there exists a hypergraph with chromatic number k, which is r-uniform, and has girth at least g. Let $\mathcal{H}$ be such a hypergraph with parameters $k = 3$, $r = 3$, and $g = 4$, which also is minimal in the sense that the removal of any hyperedge makes $\mathcal{H}$ 2-colorable. We may assume without loss of generality that $v_1v_2v_3$ is a hyperedge of $\mathcal{H}$.

Step 2: The 2-intersection graph minus $v_1v_2v_3$.

Let H be the graph on the same vertex set as $\mathcal{H}$, with two vertices being adjacent in H if and only if they are contained in a common hyperedge of $\mathcal{H}$ except for $v_1v_2v_3$. Then the vertex sets of K_3-subgraphs in H are precisely the hyperedges of $\mathcal{H}$ except $v_1v_2v_3$, because the hypergraph was supposed to have girth at least 4. In this way the K_3-WORM colorings of H with two colors are in one-to-one correspondence with the proper 2-colorings of $\mathcal{H} - v_1v_2v_3$ (by $\mathcal{H} - v_1v_2v_3$ we mean the removal of the single edge $v_1v_2v_3$, while the vertex set of the hypergraph remains unchanged). Observe further that in every K_3-WORM coloring of H the three vertices v_1, v_2, v_3 have the same color.

Step 3: Large monochromatic subsets.

Taking several vertex-disjoint copies $H_1, H_2, \ldots, H_m$ of H and in any two consecutive ones identifying v_3 of the predecessor with v_1 of the successor, we can ensure that in the graph H^* obtained, all copies of v_1, v_2, v_3 get the same color in every K_3-WORM coloring with two colors. In particular, already with $m = 3$, we can create a set of six vertices, say $S = \{x, y, x', y', x'', y''\}$, whose mutual distances are at least 3 in H^*, and S is monochromatic in every K_3-WORM coloring if just two colors are used.

Step 4: The completion.

We now take three new vertices z, z', z'' which are mutually adjacent, moreover they have degree 4 in such a way that each of $\{x, y, z\}$, $\{x', y', z'\}$, $\{x'', y'', z''\}$ induces a K_3. This needs the insertion of the additional edges $xy, x'y', x''y''$; but they do not create any further triangles because no two elements of S have any common neighbors in H^*. If S is monochromatic, then the color of S cannot occur in $\{z, z', z''\}$, but this triplet is not allowed to be monochromatic. Consequently, the graph has $W^- > 2$, proving the theorem. $\square$

5 Algorithmic complexity

In this section we consider two algorithmic problems: to determine the minimum number of colors, and to decide whether no gaps occur in the chromatic spectrum.

5.1 Lower chromatic number

Here we prove that the determination of $W^-(G, K_3)$ is NP-hard, and it remains hard even when the input is restricted to graphs with maximum degree 9. We give degree-restricted versions of such results for every number $k \geq 3$ of colors. At the end of the subsection we prove a theorem on 2-colorings, but without upper bound on vertex degrees.

More formally, we will consider the case $F = K_3$ of the following decision problem for every positive integer k.

F-WORM k-Colorability
Input: An F-WORM-colorable graph $G = (V, E)$.
Question: Is $W^-(G, F) \leq k$?

To prove the NP-completeness of this problem for $F = K_3$, we will refer to our constructions from Section 3 and the following result of Maffray and Preissmann concerning the complexity of deciding whether a graph has a proper vertex coloring with a given number k of colors, which we shall refer to as Graph k-Colorability.

Theorem 6 ([14]) (*i*) *The* Graph 3-Colorability *problem remains* NP*-complete when the input is restricted to the class of triangle-free graphs with maximum degree four.*

(*ii*) *For each* $k \geq 4$*, the* Graph k-Colorability *problem is* NP*-complete on the restricted class of triangle-free graphs with maximum degree* $3 \cdot 2^{k-1} + 2k - 2$.

By a closer look into the proof in [14] we see that this theorem is also valid if we *exclude the regular graphs* of degrees specified above. Hence attaching a pendant edge to a vertex of minimum degree we get hard problem instances of minimum degree 1, without increasing the given bounds on maximum degree.

Theorem 7 *(i) The decision problem of* K_3-WORM 3-Colorability *is* NP*-complete already on the class of graphs with maximum degree 9.*

(ii) The decision problem of K_3-WORM k-Colorability *is* NP*-complete for each* $k \geq 4$ *already on the class of graphs with maximum degree* $3 \cdot 2^k + 4k - 3$.

Proof. The problems are clearly in NP. To prove (i), we reduce the Graph 3-Colorability problem on the class of triangle-free graphs to the problem of K_3-WORM 3-Colorability. Consider a generic input graph G of the former problem with $\Delta(G) = 4$. Without loss of generality we can assume that G is connected and has a degree-1 vertex v_0. Then, we define H to be the graph $G \boxtimes K_2$, as in Section 3. Observe that $\Delta(H) = 9$. In the next step, we take three vertex-disjoint copies H^1, H^2, and H^3 of H, and make the following three identifications of vertices, each of which originates from the vertex v_0 of G:

$$x_0^1 = y_0^2, \qquad x_0^2 = y_0^3, \qquad x_0^3 = y_0^1$$

The maximum degree of the obtained graph F remains 9, as the vertices x_0^i and y_0^i had only degree 3 in H^i. By Lemma 2, and similarly to the proof of Theorem 3, we obtain that $\chi(G) = 3$ if and only if $W^-(F, K_3) = 3$. Thus, part (i) of Theorem 6 implies the NP-completeness of K_3-WORM 3-Colorability for graphs of maximum degree 9.

Part (ii) of our theorem follows from Theorem 6 (ii) by similar steps of reductions as discussed above. □

The following result states that the case of two colors is already hard.

Theorem 8 *The decision problem of* K_3-WORM 2-Colorability *is* NP*-complete on* K_3*-WORM-colorable graphs.*

Proof. We apply reduction from the 2-colorability of 3-uniform hypergraphs; we denote by $\mathcal{H} = (X, \mathcal{F})$ a generic input of this problem. Hence, X is the vertex set of $\mathcal{H}$, and $\mathcal{F}$ is a family of 3-element subsets of X. It is NP-complete to decide whether there exists a proper 2-coloring of $\mathcal{H}$, that is a partition (X_1, X_2) of X such that each $F \in \mathcal{F}$ meets both X_1 and X_2 [12].

From $\mathcal{H} = (X, \mathcal{F})$ we construct a graph $G = (V, E)$ such that $\mathcal{H}$ has a proper 2-coloring if and only if G has a K_3-WORM coloring with two colors. This correspondence between $\mathcal{H}$ and G will imply the validity of the theorem.

For each hyperedge $F \in \mathcal{F}$ of $\mathcal{H}$ and each vertex $x \in F$, we create a vertex $(x, F) \in V$ of G. If $F = \{x, x', x''\}$, then the vertices $(x, F), (x', F), (x'', F)$ will be mutually adjacent in G. Moreover, small gadgets will ensure that any two vertices $(x, F'), (x, F'') \in V$ with the same x get the same color whenever G is K_3-WORM-colored.

To ensure this, suppose that an x is incident with the hyperedges $F_1, \dots, F_d$. Then, for any two edges F_i, F_{i+1} having consecutive indices in this set (where $1 \le i < d$), we take a graph $H(x, i)$ which is isomorphic to $K_5 - e$, and identify its two non-adjacent vertices — say y and z — with (x, F_i) and (x, F_{i+1}), respectively. We make this kind of extension for each pair (x, i) in such a way that the triangles $H(x, i) - y - z$ are mutually vertex-disjoint. Let G denote the graph obtained in this way.

Consider any of the gadgets $H = H(x, i)$; we shall abbreviate it as H. Every K_3-WORM coloring of H uses a color twice on $H - y - z$, therefore the second color of $H - y - z$ (which occurs just once there) must be repeated on y and on z as well, for otherwise $H - y$ or $H - z$ would violate the conditions of K_3-WORM coloring. Thus, all of $(x, F_1), \dots, (x, F_d)$ sharing any x must have the same color. Consequently, every K_3-WORM coloring of the obtained graph G defines a proper vertex coloring of $\mathcal{H}$ in a natural way.

Conversely, if $\mathcal{H}$ is properly colored, we can assign the color of each $x \in X$ to all vertices of type (x, F) with the same x. Then, in each $H(x, i)$, the non-adjacent vertices y and z have the same color. Repeating this color on one vertex of $H(x, i) - y - z$ and assigning one different color to its remaining vertex pair we eventually obtain a K_3-WORM coloring of G. Moreover, if $\mathcal{H}$ is 2-colored, we do not need to introduce any further colors for G.

The two-way correspondence between the 2-colorings of $\mathcal{H}$ (if they exist) and the K_3-WORM colorings of G with two colors verifies the validity of the theorem. $\square$

5.2 The Chromatic Gap decision problem

The problem considered in this subsection is as follows.

F-WORM Chromatic Gap
Input: An F-WORM-colorable graph G.
Question: Does the F-WORM chromatic spectrum of G have a gap?

Here we prove:

Theorem 9 *The K_3-WORM Chromatic Gap problem is* NP*-hard.*

Proof. Part (ii) of Lemma 2 yields that the K_3-WORM chromatic spectrum of the graph $G_k \boxtimes K_2$ is gap-free if and only if G_k has a proper vertex coloring with at most three colors. This property is NP-hard (actually NP-complete) to decide. $\square$

6 3-degenerate and 4-colorable graphs

Here we show that three of the four basic problems listed in Section 2 (except the upper chromatic number) have a simple solution on 3-degenerate graphs. At the end of the section we also include an extension concerning colorability and lower chromatic number for 4-colorable graphs.

Theorem 10 *If G is a 3-degenerate graph, then*

(i) *G is K_3-WORM-colorable;*

(ii) *$W^-(G, K_3) \le 2$; and*

(iii) *G has a gap-free K_3-WORM chromatic spectrum.*

Proof. The proof proceeds by induction on the order of the graph. Consider a 3-degenerate graph G, and a vertex $v \in V(G)$ which has three neighbors, say a, b, and c. By the induction hypothesis, the graph G^- obtained by removing v and its incident edges has a K_3-WORM coloring φ which uses at most two colors, say colors 1 and 2. If $\varphi(a) = \varphi(b) = \varphi(c)$, then define $\varphi(v) = 3 - \varphi(a)$. Otherwise, the color, which occurs on exactly one vertex among a, b, and c, is assigned to v. It is easy to see that the 2-coloring obtained for G is a K_3-WORM coloring. Moreover, if the degree of v is smaller than 3, the coloring φ of G^- has a similar extension. This proves (i) and (ii).

Assume that G^- has a gap-free chromatic spectrum. We show that G has a K_3-WORM coloring with exactly t colors for each $t \ge 2$ in the range $W^-(G^-, K_3) \le t \le W^+(G^-, K_3)$. To do this, we start with a t-coloring φ of G^- and consider the neighbors a, b, and c of v. First, assume that $\varphi(a)$, $\varphi(b)$, and $\varphi(c)$ are pairwise distinct. Then abc is not a triangle. If a, b, c induce a P_3, the color of its central vertex can be repeated on v. If a, b, c induce only one edge, say ab, then $\varphi(a)$ can be assigned to v. If a, b, c are pairwise non-adjacent then v can get any of the t colors of G^-. Next, consider the case of $\varphi(a) = \varphi(b)$. If this color is different from $\varphi(c)$, then it is appropriate

to define $\varphi(v) = \varphi(c)$. In the last case, $\{a, b, c\}$ is monochromatic and v can be assigned to any color which is different from $\varphi(a)$. This proves that G is K_3-WORM colorable with exactly t colors for each t with $t \geq 2$ and $W^-(G^-, K_3) \leq t \leq W^+(G^-, K_3)$.

Note that $W^-(G, K_3) = 1$ if and only if G is triangle-free, and this implies gap-free spectrum; moreover observe that $W^+(G, K_3) \leq W^+(G^-, K_3) + 1$. By induction, we obtain that (iii) holds for every 3-degenerate graph. $\square$

Suppose now that G has maximum degree 3. By Proposition 10 we know that G is K_3-WORM-colorable, has $W^-(G, K_3) = 2$, and its chromatic spectrum is gap-free. Next, we show that $W^+(G, K_3)$ can be computed efficiently.

Let G^Δ be the graph obtained from G by removing all edges which are not contained in any triangles. This G^Δ can have the following types of connected components:

$$K_1, \qquad K_3, \qquad K_4 - e, \qquad K_4.$$

For these four types of F, let us denote by $n_G(F)$ the number of components isomorphic to F in G^Δ.

Theorem 11 *If G has n vertices, and has maximum degree at most 3, then*

$$W^+(G, K_3) = n - n_G(K_3) - n_G(K_4 - e) - 2n_G(K_4).$$

Moreover, $W^+(G, K_3)$ can be determined in $O(n)$ time.

Proof. A vertex coloring is a K_3-WORM coloring of G if and only if it is a K_3-WORM coloring of each connected component in G^Δ. Starting from the rainbow coloring of the vertex set, a K_3-WORM coloring with maximum number of colors needs:

- to decrease the number of colors from 3 to exactly 2 in a K_3 component,
- to make the pair of the two degree-3 vertices monochromatic in a $K_4 - e$ component,
- to reduce the number of colors from 4 to 2 in a K_4 component.

This proves the correctness of the formula on $W^+(G, K_3)$. Linear time bound follows from the fact that one can construct G^Δ and enumerate its components of the three relevant types in $O(n)$ steps in any graph of maximum degree at most 3. $\square$

From the formula above, the following tight lower bounds can be derived; part (*ii*) was proved for cubic graphs by Goddard et al. in [7].

Corollary 12 *If G is a graph of order n and maximum degree 3, then*

(*i*) $W^+(G, K_3) \geq n/2$, *with equality if and only if* $G \cong \frac{n}{4}K_4$;

(*ii*) *if G does not have any K_4 components, then* $W^+(G, K_3) \geq 2n/3$, *with equality if and only if G contains $\frac{n}{3}K_3$ as a subgraph;*

(*iii*) *if G does not have any K_4 components, and each of its triangles shares an edge with another triangle, then* $W^+(G, K_3) \geq 3n/4$, *with equality if and only if G contains $\frac{n}{4}(K_4 - e)$ as a subgraph.*

Proof. The formula in Theorem 11 shows that the number of colors lost, when compared to the number of vertices, is 2 from 4 in K_4, 1 from 3 in K_3, and 1 from 4 in $K_4 - e$. □

A notable particular case of (*ii*) is where $n \geq 5$ and G is connected. Moreover, since $K_4 - e$ has just two vertices of degree 2, contracting each copy of $K_4 - e$ in the extremal structure described in (*iii*) we obtain a collection of vertex-disjoint paths and cycles (where cycles of length 2 are also possible).

The complete graph K_5 shows that not every 4-degenerate graph is K_3-WORM-colorable. On the other hand, an important subclass of 5-degenerate graphs, namely planar graphs, satisfy at least the properties (*i*) and (*ii*) from Theorem 10. This was commented to us after our talk at the AGTAC 2015 conference by Kenta Ozeki. His remark inspired us to formulate also part (*ii*) of Theorem 13 below; since its proof does not require any idea beyond part (*i*), we think that the entire result should be attributed to Ozeki.

Theorem 13 *For a graph G, either of the following conditions is sufficient to ensure that G is K_3-WORM-colorable and* $W^-(G, K_3) \leq 2$ *holds:*

(*i*) *G is planar,*

(*ii*) *and more generally if G is 4-colorable.*

Proof. If (V_1, V_2, V_3, V_4) is a vertex partition of G into four independent sets, then each of $V_1 \cup V_2$ and $V_3 \cup V_4$ meets all triangles of G. This implies (*ii*), and then (*i*) follows by the Four Color Theorem. □

Remark 14 *Theorem 13(i) can also be derived by a modification of the proof of [10, Theorem 2.1], without using the 4CT. In the quoted result, Kündgen and Ramamurthi prove WORM 2-colorability of triangular* faces *of planar graphs; i.e., the condition is not required there for separating triangles.*

We thank Kenta Ozeki for inviting our attention to the paper [10].

7 Concluding remarks

We have solved several problems — some of them raised in [7] — concerning the K_3-WORM colorability and the corresponding lower chromatic number of graphs. Further properties of K_3-WORM feasible sets and the complexity of determining the upper chromatic number will be studied in the successor of this paper, [6].

Below we mention several problems which remain open. The first one proposes an extension of Theorem 5.

Conjecture 15 *For every integer $k \geq 4$ there exists a K_3-WORM-colorable K_4-free graph G such that $W^-(G, K_3) = k$.*

The other problems deal with algorithmic complexity. We have proved that it is NP-hard to test whether $\Phi_w(G, K_3)$ is gap-free. On the other hand, $n-1$ questions to an NP-oracle in parallel (asking in a non-adaptive manner whether the input graph G of order n admits a K_3-WORM coloring with exactly k colors, for $k = 2, 3, \ldots, n$) solves the problem, hence it is in the class Θ_2^p (see [13] for a nice introduction to Θ_2^p, or the last part of [1] for short comments on its properties). However, the exact status of the problem is unknown so far.

Problem 16 *Is the decision problem* K_3-WORM CHROMATIC GAP Θ_2^p-*complete?*

In the class of K_4-free graphs we do not even have a lower bound on the complexity of this problem.

Problem 17 *What is the time complexity of deciding whether the K_3-WORM chromatic spectrum of a K_3-WORM-colorable K_4-free input graph is gap-free?*

Even simpler open questions deal with the upper chromatic number.

Problem 18 *Determine the time complexity of deciding whether $W^+(G, K_3) \geq k$, where G is K_4-free,*

- *k is a given integer, or*
- *k is part of the input,*

and a K_3-WORM coloring of G with fewer than k colors is given in the input.

Also, the classes of d-degenerate graphs for various values of d offer interesting questions.

Problem 19 (i) *Can the value of $W^+(G, K_3)$ be determined in polynomial time on 3-degenerate graphs?*

(ii) *If the answer is yes, what is the smallest d such that the computation of $W^+(G, K_3)$ is NP-hard on the class of d-degenerate graphs?*

(iii) *Prove that a finite threshold value d with the property described in part (ii) exists.*

Problem 20 *Consider the class of graphs with maximum degree at most d.*

(i) *Is it NP-complete to decide whether $W^-(G, K_3) = 2$ if d is large enough?*

(ii) *What is the smallest d_k as a function of k such that the decision of $W^-(G, K_3) \leq k$ is NP-hard on the class of graphs with maximum degree d_k?*

(iii) *What is the smallest d for which it is NP-complete to decide whether a generic input graph of maximum degree at most d is K_3-WORM colorable?*

Finally, a very natural and general problem is:

Problem 21 *Investigate the analogous problems for graphs F other than K_3.*

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